which expresses the probability w that a molecule, leaving the wall at position $X = x/l$ from the inlet, will reach the outlet of the tube.

The limiting case of the infinitely long tube was analyzed separately, because a direct numerical integration of Eq. (2) proved increasingly inaccurate as $D \to 0$. It can be shown that[3]

$$\frac{p_{t\infty}}{p_1}\sqrt{\frac{T_1}{T_g}} = e^{-S^2} + \frac{\sqrt{\pi}S}{2}\left(1 + \text{erf } S - \frac{6}{\pi}\lim_{D\to 0}\frac{d\mu}{dD}\right) \qquad (4)$$

where

$$\lim_{D\to 0}\frac{d\mu}{dD} = \int_0^{\infty}\frac{\eta(S,D/X) - \eta(S,O)}{(D/X)^2}\,d(D/X) \qquad (5)$$

To make Eq. (5) suitable for numerical integration, the approximation

$$\lim_{D\to 0}\frac{d\mu}{dD} = -\int_0^{(D/X)_I}\frac{\frac{\pi}{4}(1 + \text{erf } S) - \eta(S,D/X)}{(D/X)^2}\,d(D/X) - \frac{\pi(1 + \text{erf } S)}{4(D/X)_I} \qquad (6)$$

proves useful, since the integrand in this equation becomes a known function[3] of S at $D/X = 0$. The accuracy attainable with Eq. (6) increases with increasing values of $(D/X)_I$. Fig. 1 shows the calculated results for the infinitely long tube using $(D/X)_I = 220$.

For convenience of representing the data, the quantity $p_t\sqrt{T_1}/p_1\sqrt{T_g}$ has been divided by the orifice function $\chi(S) = p_0\sqrt{T_1}/p_1\sqrt{T_g}$. The portion of the resulting ratio which is above unity, $(p_t - p_0)/p_0$, has been plotted in Fig. 1.

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Pure Bending, Twisting, and Stretching of Skewed, Heterogeneous, Aeolotropic Plates

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The following discussion is concerned with the problems of pure bending, twisting, and stretching in the theory of heterogeneous aeolotropic plates. Previous results[1] for rectangular laminated plates will be extended to skewed heterogeneous plates.

Formulation of the Problem

Consider a thin elastic plate of constant thickness that is heterogeneous in the thickness direction z. Let x,y be the coordinates in the undeflected bottom face of the plate ($z = 0$). The general theory of bending and stretching of such a plate was established in Refs. 1 and 2 and is used in the present development.

The expressions for stress couples in terms of stress resultants and plate curvatures κ are of the form

$$[M] = [C^*][N] + [D^*][\kappa] \qquad (1)$$

where the C^*, D^* matrices are defined by Eq. (14) of Ref. 2.

We consider heterogeneous plates bounded by two straight lines $x = \pm 1$ and by two straight lines $y = \pm{}^1/_2c - xtg\alpha$. Angle α is the angle of skew of the plate, 21 is the span of the plate, and c is the chord of the plate.

Expressions for bending moment M_n and twisting moment M_{nt} acting along the edges $y = \pm{}^1/_2c - xtg\alpha$ follow by means of the usual transformation equations for plate stress couples, which hold for homogeneous, as well as heterogeneous, plates:

$$M_n = M_y\cos^2\alpha + M_x\sin^2\alpha + 2M_{xy}\cos\alpha\sin\alpha \qquad (2)$$

$$M_{nt} = (M_x - M_y)\cos\alpha\sin\alpha + M_{xy}(\cos^2\alpha - \sin^2\alpha) \qquad (3)$$

Within the framework of this theory,[1, 2] which does not account for transverse effects, there occur concentrated forces P at the corners of the plate given by

$$P = (M_{xy} + M_{nt})_{corner} \qquad (4)$$

It was shown in Ref. 1 that for the problems of uniform distributions of stress resultants and couples, with which we are concerned, it is sufficiently general to assume a deflection function

$$w = {}^1/_2k_xx + {}^1/_2k_yy + k_{xy}xy \qquad (5)$$

where k_x, k_y, k_{xy} are the constant values of the plate curvatures κ_x, κ_y, κ_{xy}, respectively.

Pure Bending of Heterogeneous Skewed Plate

In this case the following boundary conditions must be satisfied:

$$\text{at } x = \pm 1: cM_x = M_1,\ N_x = 0,\ N_{xy} = 0 \qquad (6)$$

$$\text{at } y = \pm{}^1/_2c - xtg\alpha:\ (2l/\cos\alpha)M_n = M_2,\ N_n = 0,\ N_{nt} = 0 \qquad (7)$$

$$\text{at the corners: } M_{xy} + M_{nt} = 0 \qquad (8)$$

Using Eqs. (1)–(3) and (6)–(8), we find for the constants in Eq. (5) the following system of three algebraic equations:

$$\begin{bmatrix}\mathfrak{D}_{11} & \mathfrak{D}_{12} & \mathfrak{D}_{13}\\ \mathfrak{D}_{21} & \mathfrak{D}_{22} & \mathfrak{D}_{23}\\ \mathfrak{D}_{31} & \mathfrak{D}_{32} & \mathfrak{D}_{33}\end{bmatrix}\begin{bmatrix}k_x\\ k_y\\ k_{xy}\end{bmatrix} = \begin{bmatrix}M_x\\ M_n\\ 0\end{bmatrix} \qquad (9)$$

where

$$\mathfrak{D}_{11} = D_{xx}^*,\ \mathfrak{D}_{12} = D_{xy}^*,\ \mathfrak{D}_{13} = D_{xs}^* \qquad (10)$$

$$(\mathfrak{D}_{21}, \mathfrak{D}_{22}, \mathfrak{D}_{23}) = (D_{xy}^*, D_{yy}^*, D_{ys}^*)\cos^2\alpha + (D_{xx}^*, D_{xy}^*, D_{xs}^*)\sin^2\alpha + 2(D_{xs}^*, D_{ys}^*, D_{ss}^*)\cos\alpha\sin\alpha \qquad (11)$$

$$(\mathfrak{D}_{31}, \mathfrak{D}_{32}, \mathfrak{D}_{33}) = ([D_{xx}^* - D_{xy}^*], [D_{xy}^* - D_{yy}^*], [D_{xs}^* - D_{ys}^*])\cos\alpha\sin\alpha + 2(D_{xs}^*, D_{ys}^*, D_{ss}^*)\cos^2\alpha \qquad (12)$$

The solution of Eqs. (9) is compactly written as

$$[k] = [\mathfrak{D}^{-1}][M] \qquad (13)$$

Using Eqs. (5) and (13) together with Eq. (1), and noting that all stress resultants vanish throughout the plate, we obtain the couples expressed in the x-y coordinates. Eqs. (61) and (70) of Ref. 1 give, then, the components of strain and stress, respectively.

Twisting of Heterogeneous Skewed Plates

The following boundary conditions prevail:

$$\text{at } x = \pm 1: M_x = 0,\ N_x = 0,\ N_{xy} = 0 \qquad (14)$$

$$\text{at } y = \pm{}^1/_2c - xtg\alpha: M_n = 0,\ N_n = 0,\ N_{nt} = 0 \qquad (15)$$

In addition to this, we have the applied torque given by

$$T = cP \qquad (16)$$

Following the same procedure as above, we find for the constants in Eq. (5)

$$[k] = [\mathfrak{D}^{-1}]\begin{bmatrix}0\\ 0\\ T/c\end{bmatrix} \qquad (17)$$

The method for determining stresses and strains is exactly as indicated in the case of pure bending.

Stretching of Heterogeneous Skewed Plates

The boundary conditions to be considered now are

$$\text{at } x = \pm 1: \quad M_x = 0,\ N_x = 0,\ N_{xy} = 0 \qquad (18)$$

$$\text{at } y = \pm {}^1/_2 c - x tg\alpha: \quad M_n = 0, 21N_y = N,\ N_{xy} = 0 \qquad (19)$$

$$\text{at the corners: } \quad M_{xy} + M_{nt} = 0 \qquad (20)$$

Using the same procedure as before, we find for the constants in Eq. (5)

$$[k] = -N_y[\mathfrak{D}^{-1}] \times \begin{bmatrix} C_{xy}^* \\ C_{yy}^* \cos^2\alpha + C_{xy}^* \sin^2\alpha + 2C_{ys}^* \cos\alpha \sin\alpha \\ C_{ys}^* + (C_{xy}^* - C_{yy}^*)\cos\alpha \sin\alpha + C_{ss}^*(\cos^2\alpha - \sin^2\alpha) \end{bmatrix} \qquad (21)$$

Thus simple stretching of the plate is associated with a deflection pattern [Eqs. (5) and (21)]. Present results for pure bending and twisting include—as a special case previously obtained—expressions by Reissner[3] for homogeneous, isotropic, skewed plates.

It is noted that the solution [Eq. (21)] for the stretching problem and the cross effect in the stress-resultants-couples relations, observed in all three cases analyzed, do not appear at all in homogeneous plates.

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Slip Flow of an Ionized Gas Over a Sphere Carrying a Magnetic Dipole†

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In this note, the work of Barthel and Lykoudis[1] on the motion of a conducting gas about a magnetized sphere at low Reynolds number is extended to the case in which the effects of slip flow become important.

The ordinary magnetofluidmechanic approximations are used throughout; hence, the Navier-Stokes equations, with the added terms representing the ponderomotive force, are used. It is open to question whether the Navier-Stokes equations are valid at low densities where slip phenomena are important. However, there are enormous difficulties, as well as serious ambiguities, connected with the application of the Burnett, thirteen-moment, and other equations.[2]

The effects of slip are accounted for, following Maxwell, by assuming that the tangential velocity is proportional to the velocity gradient at the wall. The coefficient of proportionality is designated as the slip coefficient and is given the symbol ζ. It is related to the mean free path by the equation

$$\zeta = [(2 - \epsilon)/\epsilon]l$$

where l is the mean free path and ϵ is the fraction of molecules which are diffusely reflected from the wall (with zero average tangential velocity).

The problem of slip flow over a nonmagnetized sphere at low Reynolds numbers was solved by Basset;[3] therefore, his solution is used in calculating the ponderomotive force in the perturbation analysis similar to that applied in Ref. 1.* Small magnetic Reynolds numbers are assumed; hence, the undistorted magnetic field for a dipole is used in the calculation of the ponderomotive force.

Applying the same method of solution as used in Ref. 1, the result for the coefficient of drag on the sphere due to the viscous forces, pressure forces, and the ponderomotive force is**

$$C_D = C_{D_0}\left[1 + M\left(\frac{0.004 + 0.170\phi + 1.737\phi^2 + 5.879\phi^3 + 6.281\phi^4}{0.573 + 8.975\phi + 47.429\phi^2 + 103.829\phi^3 + 80.571\phi^4}\right)\right] \qquad (1)$$

where

$$\phi = \zeta/a = [(2 - \epsilon)/\epsilon](l/a)$$

and $M = \sigma B_0^2 a^2/\mu$ is the square of the usual Hartmann number, σ the electrical conductivity, B_0 the magnetic flux at the poles, a the radius of the sphere, and μ the dynamic viscosity.

In the nonmagnetic case C_{D_0} represents the drag coefficient allowing for slip and is given by

$$C_{D_0} = (12/R)[(1 + 2\phi)/(1 + 3\phi)]$$

where R is the kinematic Reynolds number. When $\phi = 0$ (no slip), Eq. (1) reduces to

$$C_D = (12/R)[1 + (M/150)]$$

which concurs with the first-iteration result of Ref. 1.

The figure shows the variation of C_D/C_{D_0} with ϕ (essentially the Knudsen number) at different Hartmann numbers. It can be readily seen that, for large values of the slip coefficient, the effect of the magnetic field upon drag is quite pronounced. This is due to the fact that, when appreciable velocities are allowed at the surface of the sphere, the ponderomotive forces in the vicinity of the surface can become quite large, since the magnetic field is strongest there.

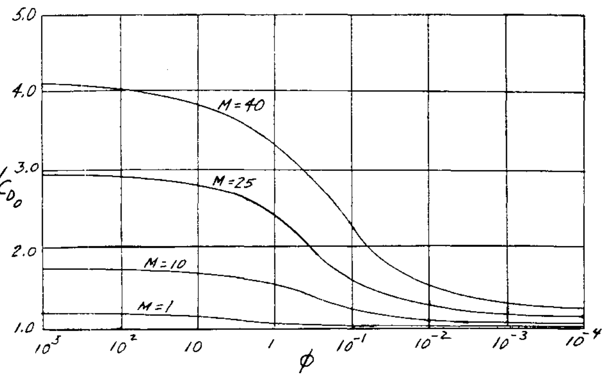

Fig. 1. Variation of drag with slip coefficient.

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** The details of these and all other calculations can be found in Ref. 4.